

Quantum Geometry from the Formalism of Loop Quantum Gravity

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Abstract

Introducing Quantum Geometry as a consequence of the quantisation procedure of loop quantum gravity. By recasting general relativity in terms of $\frac{1}{2}$ -flat connections, specified by the Holst's Modification to the Palatini action, we can recast general relativity as a gauge theory. By performing a 3+1 split of space-time a Legendre transformation can be performed to give an expression of the Hamiltonian, in terms of the configuration variables, and the Vector, Gauss and Hamiltonian constraints on it. The system is then quantised by refined algebraic quantisation; a suitable choice of polarisation is chosen and by smearing the phase space variables a suitable representation of the Poisson algebra is constructed. Lattice gauge theory is applied on graphs and the quantum configuration space and the Hilbert space, which admits a spin decomposition, is constructed. The conjugate momentum defines an operator which, its representation, defines the Kinematical Hilbert space. The constraint operators, that generate symmetries and diffeomorphism, are constructed on the kinematical Hilbert space. The geometrical operators are constructed and their corresponding eigenvalues approximate to the continuum theory.

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1 Introduction

1.1 Two Great Theories

An artist on a stationary and stable backdrop works to produce an elegant work of art. Well what if the backdrop was no longer stationary but moving, accelerating; what if the back drop was no longer stable but changing with each brush stroke. How then is the picture that the artist draws different to what we see? Now given the complete picture it becomes very difficult to trace back to see what strokes the artist made. This is the great challenge that physicist face today, tracing back from the big picture to the very fundamental forces that make it, every day physicist around the world step closer and closer to a grand unified theory.

1.1.1 Geometry

In standard Euclidean geometry we work with a flat stable background, with forces, this gives rise to classical physics, and more importantly Newtons idea of gravity. Everything then becomes set and predictable; we can observe the motion of the moon around the earth and the earth around the sun. However we can introduce a single photon into the system and observe the unexplainable, a photon does not have to travel in a straight line in Euclidean geometry. The idea to expand outside Euclidean geometry came around the mid-19th century with the work of Bernhard Riemann who suggested that the background space need not be flat. He managed to generalise geometry on to any manifold and thus lay the ground work for Einstein to formulate his theory.

His use of manifolds introduced the notion of differential geometry in order to relate a point on the overlap of two charts with different coordinates. This opened a whole new branch of mathematics. Differential forms defined by Elie Cartan are used to express differential equations independent of the coordinates and it also

carries to manifolds the notion of gradient, divergence and curl. A differential form is an anti-symmetric tensor of the type $(0, r)$, a 0-form is a scalar field and then depending on its degree, r , an r -th order differentiable equation. Then the calculus of differentiable forms, or exterior calculus, can be used to manipulate forms without the use of a metric. Then the exterior derivative can be defined on a form increasing the form to an $r + 1$ differentiable equation. Then the exterior derivative of a 0-form yields a vector and exterior derivative of a 1-form yields a tensor. This can be related to vector calculus because the exterior derivative of a 1-form is the curl of a vector and the exterior derivative of a 0-form is the gradient of the function (scalar vector).

1.1.2 General Relativity

General relativity changes the view of Newtonian gravity; it enables the mathematical description of gravity and its effects in the universe. By doing so it can describe the motions of the planets, by accurately calculating the shortest distance from A to B in curved space-time, rather than the circular motion describe by Newtonian mechanics[3]. This required Einstein to view geometry in a different way without a flat stable background, all of a sudden the world that we live in does not appear to be as flat as we first thought. However this also begs the question if the space we live in is curved then how accurate is π because the paper we draw the circle is not flat. It is like drawing a circle on a ball or on the inside of a ball and calculating the value of π . We get round this problem because the space-time is endowed with a manifold structure, more specifically a Riemannian manifold, and as a result locally the manifold may look like Euclidean space but globally it may not.

It was Riemann who first suggested that the geometry of space may have a more direct physical meaning, which general relativity is a very good example of this[1]. Matter curves space-time, it mere presence changes the area around it. *Matter tells*

space-time how to bend and space-time tells matter how to move - John Wheeler. This lead Einstein to regard gravity not as a force but as a manifestation of space-time geometry. It is in fact that gravity is geometry. Classical general relativity uses the language of Riemannian geometry to describe the kinematics of a system [4]. However as we attempt to combine general relativity with quantum mechanics a new type of language much be used, one that incorporates the curvature of space-time and its quantum effect.

1.1.3 Quantum Mechanics

Quantum theory first came about when Max Planck published his formula for the distribution of energy as a function of frequency in black body radiation, in which a new fundamental constant was introduced - Planck's Constant, h . This meant that energy came in discrete packets and each packet of energy had a frequency. At the time energy was thought to be continuous and now we find that it quantised, which manifest itself in the absorption and emission of matter, hence Quantum Mechanics.[5]

Since then it has been implemented in everyday life from electronic equipment such as macro processors to flat screen televisions. It has enabled us to go from the wave like nature of light to the photon, and it works both ways. We can take a discrete system and approximate it to a continuous system, which is easily shown by the wave like nature of electrons. However due to the wave like nature of the electron it would make it difficult to confine it to a specific volume in space, even from a particle point of view this is still the case. If we confine an electron in a region of space¹ and attempted to know its exact location by the emission/absorption of a photon by the electron. We find that the electron will either recoil due to the emission/absorption of the photon or change it energy level because of it, therefore

¹By localising the electron in an infinite potential well.

up to a point we can no longer be certain of its position or its momentum. Heisenberg's Uncertainty principle exists between any non-commuting variables, not just for position and momentum but also time and energy; if the electron does recoil then we can find out how much if we know the time of emission and the energy of the photon. The main advantage of this is quantum tunneling where the electron can 'borrow' an amount of energy for a specific amount of time in order to 'tunnel' through a potential well.

1.2 Quantum Field Theory

At the time quantum theory and general relativity were both taking off and it was soon realised that two theories were very much incompatible. On one hand you have the elegance of general relativity and on the other you have the chaotic world of quantum theory, general relativity depends on continuous variables and quantum mechanics depends on discrete variables.

In order to begin combining these two theories² we must first incorporate Special Relativity and Quantum Mechanics, the result is Quantum Field Theory[6]. This union has many interesting effects; $E = mc^2$ allows particle annihilation and creation and Heisenberg's uncertainty principle allows creation of particles for a short period of time in a vacuum.

Quantum field theory has allowed us to delve deeper into the atoms that make up the universe and helps us accurately observe, predict and explain the outcomes in a particle accelerator. It can successfully explain the fundamental forces of nature except gravity. This is mainly due to the coupling constant having negative dimensions and therefore non-renormalisable [7]. Quantum field theory also explains why the electromagnetic force is both attractive and repulsive and why gravity is only attractive.

²General Relativity and Quantum Mechanics

1.2.1 Why Masses Attract

Attraction and repulsion in quantum electrodynamics is a result of charge conservation, $\partial^\mu J_\mu(x) = 0$ the effective action is written as ³:

$$S = \frac{1}{2} \int \frac{(d^4k)}{(2\pi)^4} J^\mu(k)^* \frac{(-g_{\mu\nu} + (k_\mu k_\nu)/m^2)}{(k^2 - m^2 + i\epsilon)} J_\nu(k) \quad (1.1)$$

With charge conservation it becomes ⁴:

$$S = \frac{1}{2} \int \frac{(d^4k)}{(2\pi)^4} J^\mu(k)^* \frac{1}{(k^2 + i\epsilon)} J_\mu(k) \quad (1.2)$$

Now the potential energy between like charges is positive and hence why like charges repel. If we write the current as, $J^\mu = J_+^\mu - J_-^\mu$, then the effective becomes negative which means that the potential between opposite charges is negative and hence opposite charges attract. [6]

The stress-energy tensor, which describes how space-time is disturbed at a point, is the gravitational equivalent of the current in electromagnetism. Since this is a second rank tensor it means the Graviton must be a massless spin 2 boson. The effective action of a massive spin 2 vector boson is:

$$S = -\frac{1}{2} \int \frac{(d^4k)}{(2\pi)^4} T^{\mu\nu}(k)^* \frac{(G_{\mu\lambda}G_{\nu\sigma} + G_{\mu\sigma}G_{\nu\lambda}) - \frac{2}{3}G_{\mu\nu}G_{\lambda\sigma}}{(k^2 - m^2 + i\epsilon)} T_{\mu\nu}(k) \quad (1.3)$$

As before we impose the conservation of energy and momentum by

$\partial_\mu T^{\mu\nu}(x) = 0$ and we get:

$$S = \frac{1}{2} \int \frac{(d^4k)}{(2\pi)^4} T^{00}(k)^* \frac{(1 + 1) - \frac{2}{3}}{(k^2 - m^2 + i\epsilon)} T_{00}(k) \quad (1.4)$$

³The derivation used in [6] is for a massive vector boson were mass will be set to zero at the end.

⁴W(J)instead of S in [6]

Which is always greater than 1 and therefore the potential will always be negative and hence masses will attract and as a result why space-time only has positive curvature.[6]

1.2.2 Inconsistencies

The inconsistencies in general relativity are due to its prediction of space-time singularities, black holes and big bang singularities. Where the density of matter and the curvature of space-time tend to infinity, such infinities have no room in a fundamental theory and hence general relativity is seen as a classical approximation to a unified theory. Hence the need for a quantum theory of gravity as it could avoid these singularities in a similar way as quantum mechanics explains the stability of atoms.[17]

The main inconsistency of the quantum theory arises from the Einstein equations,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}(g) \quad (1.5)$$

where geometry and matter are coupled together. The energy-momentum tensor is governed by quantum field theory due to vacuum fluctuations, and as a result as we move in to the quantum world this is replaced by the vacuum expectation value, $\hat{T}_{\mu\nu}(g_0)$ ⁵. Then the inconsistency arises due to the dependence of the energy-momentum tensor on the metric, because $g_0 \neq g$. As a result it becomes almost impossible to obtain a field theory when the metric is dynamical even more so when the metric is dependent on the field itself. Therefore we are forced to consider background independent theories or by summing over all possible backgrounds⁶. However the resulting theory is non-renormalisable and is no longer background independent, which is the ultimate goal of the theory.

⁵The operator is one acting on a Hilbert space hence expectation values are taken.

⁶the total metric is a sum of the Mankowski metric and the perturbations of gravity[17]

1.3 The Continuum Nature of Physics

At the macroscopic level and above we find that quantisation of parameters is not necessary since the discrete steps are many and close together, so a continuous approximation is much better suited, but does not work in the quantum world[1]. Since the birth of Quantum mechanics Planck's constant has been the major factor in quantisation, it has been used to describe the smallest energy packet per second. However when combined with other fundamental constants it produces a new constant called the Planck length⁷,

$$\ell_P = \sqrt{\frac{\hbar G}{c^3}} \approx 10^{-33} \text{cm} \quad (1.6)$$

The Planck length current is conjectured to be the smallest distance measurable however the obvious question arises 'What is the physics like below the Planck length?' So far the physics well above this scale is well tested but as the physics approaches the Planck length quantisation is required. Quantisation of energy has been successful however it is necessary to quantise space-time? From Heisenberg's uncertainty relation we can see that physically there is a limit to how much we can measure distances and momentum, this is ultimately due to the Planck's constant. So what happens then between the intervals of the Planck length? We could try to probe this region with a highly energetic photon but due to quantum indeterminacy this will yield no information, the photon that energetic would create a Planck particle⁸. As a result it would be unable to escape the gravitational pull of the Planck particle. Hence to understand the dynamics at the Planck length we must first understand the geometry of space-time at this length.[25]

⁷ \hbar is $\frac{h}{2\pi}$, G is Newton's gravitational constant and c is the speed of light.

⁸Where the Compton wavelength of the particle is roughly equal to the Schwarzschild radius

2 Quantum Geometry

2.1 Why is Quantum Geometry?

The impact that general relativity has made has completely changed our concept of gravity; from Newton's idea about a physical interacting planets to Einstein's theory of mass causing curvature. One of the main point of general relativity is that gravity is geometry; it is no longer just the interaction of two masses attracting but the motion due to the curvature of space-time. This opens the philosophical debate on whether matter exist due to the curvature of space-time or does matter curve space-time.

General relativity states that the geometry of space-time is dynamical and as a result its formalism was due to a specific formalism of quantum gravity, loop quantum gravity. From the equation (1.5), attempts to quantise the energy-momentum tensor has resulted in an ambiguous metric, however we can use the operator versions of the geometrical tensors and this yields the quantum Einstein equations.

$$\hat{R}_{\mu\nu} - \frac{1}{2}\hat{R}\hat{g}_{\mu\nu} = \kappa\hat{T}_{\mu\nu}(\hat{g}) \quad (2.1)$$

where the left hand side is the quantisation of geometry and the right hand side is the quantisation of matter.

2.2 Approach to Canonical Quantum Gravity

The formalism of quantum geometry was a bi-product of the loop quantum gravity theory, it began with Ashtekar reformulating general relativity in terms of connections, more specifically half-flat connections. This meant that general relativity can then be approximated to a gauge theory and therefore attempt quantisation by construction of the Hamiltonian system, where the variables are the tetrads and

connections. The Hamiltonian can be derived from a Legendre transformation and then the system can be quantised by the process of refined algebraic quantisation. The quantisation process involves construction of the Poisson sub-algebra, which is constructed on a lattice type structure using graphs. Then we consider its representations on the kinematical Hilbert space, we find that there is a unique decomposition of the Hilbert space due to the use of the half-flat connections. The spin network decomposition arises due to the decomposition of the kinematical Hilbert space as a sum of Hilbert spaces of each lattice or graph⁹. Then the construction and regularisation of the geometrical operators can be done on the kinematical Hilbert space. Then we can impose the constraint of the Hamiltonian system on these operators and construct the dynamics of the system.

Quantum geometry, or more accurately quantum Riemannian geometry, is the mathematical framework of quantum gravity just as differential geometry provides a basic mathematical framework to formulate modern gravity theories. Quantum geometry provides the necessary techniques in the quantum domain. [26]

⁹If we consider many charts on a manifold then each chart can locally be approximated to flat space or in our case an individual lattice/graph. Then by considering the union of all charts we can generate a complete picture of the manifold

3 The Connection Theory of Relativity

3.1 Metrics and Connections

General relativity is the only fundamental force that has a dependence on the metric and hence the connection. As a result it makes sense to recast general relativity so that the metric and the connection become the basic variables in the theory. We find that general relativity as a theory of connections actually resembles a gauge theory and the description of gravitational interactions is similar to the other fundamental forces of nature. This theory was only discovered recently, in the last 25 years, however Einstein and Schrödinger both gave the reformulation many years earlier but then later abandon the idea due to a more complicated basic equations and a more complicated metric.[1]

Riemann geometry, the language of general relativity, uses the metric and connection to define distance and curvature within a manifold. In the specific case of general relativity the usual notions of defining distance change hence the need to have a metric and a connection. The connection is defined to be the parallel transport of the objects¹⁰ along a curve but in general if the curve is closed, i.e. a loop, then the object does not return to its original position. In fact it is rotated by a unitary matrix, where this unitary matrix is a measure of curvature in the region enclosed by the closed curve. [1] Therefore we can use the connection as an independent variable rather than before where we used the metric which gives a connection, the Levi-Civita connection. In the case that a Riemannian manifold has vanishing torsion¹¹ we obtain classical Newtonian gravity, the metric connection then becomes the Levi-Civita connection¹².

¹⁰vectors, tensors...etc.

¹¹Torsion is the antisymmetric part of the connection and characterises the twisting of the tetrads along a curve.

¹²This is just a symmetric part of the connection[12]

3.2 Palatini Action and the Symplectic Structure of Phase Space

The Palatini framework assumes that the metric and connection are basic and independent and the corresponding Palatini action writes the traditional Einstein-Hilbert action in terms of tetrads and differential forms. This makes it easier when performing differential calculations on a manifold as we can take advantage of the background independence of the differential forms. We begin by defining a 4-dimensional manifold \mathcal{M} with a fixed orientation and a system of one-forms that provide a basis for the co-tangent space, these basis one-forms are known as co-tetrads or co-frame fields¹³.

The Palatini action is given by,

$$S_{(p)}(e_\mu^I, \omega_{\mu J}^I) = \frac{1}{4\kappa} \int_{\mathcal{M}} \epsilon_{IJKL} e_\mu^I \wedge e_\nu^J \wedge \Omega_\mu^{KL} \quad (3.1)$$

where $\kappa = 8\pi G$ and ϵ_{IJKL} defines the orientation of the manifold. The co-tetrads, e_μ^I determine the metric, $g_{\mu\nu} = \eta_{IJ} e_\mu^I e_\nu^J$ where η_{IJ} is the fixed metric¹⁴. Ω^{KL} is the curvature 2-form defined by the second cartan structure equation,

$$\Omega_{\mu J}^I = d\omega_{\mu J}^I + \omega_{\mu K}^I \wedge \omega_{\mu J}^K \quad (3.2)$$

where $\omega_{\mu J}^I$ is the spin connected 1-form. In the case of vanishing Torsion the first cartan structure becomes,

$$0 = de_\mu^I + \omega_{\mu J}^I \wedge e_\mu^J \quad (3.3)$$

¹³ I, J, K, \dots represent the internal indices of the tetrad and μ, ν, \dots represent the space-time indices

¹⁴Lorentz metric $\left\{ \begin{array}{cccc} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right\}$, or Riemann metric, $\left\{ \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right\}$.

which implies that $\omega_{\mu J}^I$ is dependent on e_{μ}^I , the co-tetrad.

We can reduce the Palatini action to the more familiar Einstein-Hilbert action by going from the non-coordinate basis to the coordinate ones.

$$S_{(H)}(e_{\mu}^I, \omega_{\mu J}^I(e)) = \frac{1}{2\kappa} \int_{\mathcal{M}} \sqrt{-\det(g)} d^4x g^{\mu\nu} R_{\mu\nu} \quad (3.4)$$

where $g^{\mu\nu} R_{\mu\nu} = \mathcal{R}$ is the Ricci Scalar and $\sqrt{-\det(g)} d^4x = \epsilon_{IJKL} e_{\mu}^I \wedge e_{\nu}^J$ which is the volume element¹⁵.

Holst modified version of the Palatini action (3.1) results in the configuration variable being real. When Ashtekar presented his variables they made the Hamiltonian constraint much simpler than that of the AMD method, however this introduced a complex phase space action. Further work by Samuel, Jacobson and Smolin resulted in Barbero theorising and defining new variables based on Ashtekar's ones. Unfortunately this did not work, but it lead Holst to follow Barbero formalism and modify the action by solving for the rotational part of the spin-connection[8]. The actual modification is very similar to that of the modification to the action in the Yang-Mills theory, and as such the addition of the extra term does not change the equations of motion. Also the actual Holst modification is always zero because it satisfies the Cartan structure equation and due to vanishing Torsion it will therefore yield the Einstein-Hilbert action.[4]

The action with Holst's modification is give by,

$$S_{(H)} = S_{(P)} - \frac{1}{2\kappa\gamma} \int_{\mathcal{M}} e^I \wedge e^J \wedge \Omega_{IJ} \quad (3.5)$$

where γ is the Barbero-Immirzi¹⁶. We also find that γ is analogous to the θ -

¹⁵The coordinate differentials, $dx, dy \dots$ etc. are usually held within the differential forms themselves.

¹⁶in [8] parameter and for applications in quantum theory γ is non-zero this is given as α and in [9] it is given as $\frac{1}{\beta}$

parameter in quantum chromodynamics. the term in the Palatini action Ω^{IJ} is the dual operator of Ω_{IJ} and hence defined by,

$$\gamma\Omega^{IJ} = \gamma * \Omega_{IJ} = \frac{\gamma}{2}\epsilon_{IJ}^{KL}\Omega_{KL} \quad (3.6)$$

We continue by defining a symplectic structure¹⁷ of the covariant phase space, this will enable us to define a half-flat connection. The symplectic form, Ω , is a closed differentiable 2-form that is non-degenerate¹⁸.^[12] A symplectic 2-form can expressed as the exterior derivative of a one-form, θ ,

$$\theta = q_\mu dp^\mu \quad (3.7)$$

which yields the following properties of Ω .

$$d\Omega = d^2\theta = 0 \quad (3.8)$$

$$\Omega_p(X_p, Y_p) = 0 \quad \forall Y_p \in T_p(\mathcal{M}) \Rightarrow X_p = 0 \quad (3.9)$$

in [13] we find that the symplectic one-form, θ , is defined through the calculations of variations on the Lagrangian, since, $\mathcal{L} = \mathcal{L}(e, A)$, then the symplectic one-form is given by:

$$\theta = \int_M \frac{\partial \mathcal{L}}{\partial \omega_{\mu J}^I} \wedge de_\mu^I \quad (3.10)$$

and using the equation (3.8) the symplectic two-form is defined by,

$$\Omega = d\left(\int_M \frac{\partial \mathcal{L}}{\partial \omega_{\mu J}^I} \wedge de_\mu^I\right) \quad (3.11)$$

¹⁷the generalises the Hamiltonian description of classical systems there by making it coordinate independent.

¹⁸In classical mechanics it is given by the cartan wedge product between the exterior derivative of the phase space variables $dq \wedge dp$

using the formula,

$$d(\chi \wedge d\omega) = d\chi \wedge d\omega + (-1)^q \chi \wedge d^2\omega^{19} \quad (3.12)$$

and the property from (3.8) we find that the symplectic two-form becomes,

$$\Omega = \int_M d \left[\frac{\partial \mathcal{L}}{\partial \omega_{\mu J}^I} \right] \wedge de_{\mu}^I \quad (3.13)$$

Therefore the variables in the phase space will be based on the co-tetrad and $\frac{\partial \mathcal{L}}{\partial \omega_{\mu J}^I}$.

Using this and (3.6) the definition of $\frac{1}{2}$ -flat connections can be used to construct a background independent theory.

3.3 $\frac{1}{2}$ Flat Connections

The variables in the phase space now become the tetrad, e_{μ}^I , and the $\frac{1}{2}$ flat connection, $\omega_{IJ}^{(+)}$, which is defined as,

$$\omega_{IJ}^{(+)} = \frac{1}{2} \omega_{IJ} - \gamma * \omega_{IJ} \quad (3.14)$$

Now $\omega_{IJ}^{(+)}$ is the anti-self dual²⁰ part of ω_{IJ} . We can also define $\Sigma_{(+)}^{IJ}$ to be the (anti)-self dual part of $e^I \wedge e^J$,

$$\Sigma_{(+)}^{IJ} = \frac{1}{2} (e^I \wedge e^J - \gamma * e_K \wedge e_l) \quad (3.15)$$

The Holst action then simplifies down to,

$$S_{(H)}(e^I, \omega_{IJ}^{(+)}) = -\frac{1}{\kappa\gamma} \int_{\mathcal{M}} \Sigma_{(+)}^{IJ} \wedge \Omega_{IJ}^{(+)} \quad (3.16)$$

¹⁹ χ is a q -form

²⁰self dual if $\gamma = -1$ and anti-self dual if $\gamma = 1$

where $\Omega_{IJ}^{(+)}$ is the curvature of $\omega_{IJ}^{(+)}$ and defined as,

$$\Omega_{\mu J}^{I(+)} = d\omega_{\mu J}^{I(+)} + \omega_{\mu K}^{I(+)} \wedge \omega_{\mu J}^{K(+)} \quad (3.17)$$

The phase space, Γ , is the Hamiltonian phase space of the theory and as such it has a specific definition of the symplectic two-form,

$$\mathbf{\Omega} = dp_\mu \wedge dq^\mu \quad (3.18)$$

which implies that our definition of the symplectic two-form can be written in term of the phase space variables (P_{IJ}^a, A^{IJ}) ²¹. Hence,

$$\mathbf{\Omega} = \int_M dP_{IJ}^a \wedge dA_a^{IJ} \quad (3.19)$$

we can define in terms of co-tetrads and $\frac{1}{2}$ -flat connections,

$$\mathbf{\Omega} = -\frac{2}{\kappa\gamma} \int_M d\Sigma_{(+)}^{IJ} \wedge d\omega_{IJ}^{(+)} \quad (3.20)$$

where A is defined as the pull back of ω and p is defined as,

$$P_{IJ}^a = -\frac{1}{2\kappa\gamma} \eta^{abc} \Sigma_{bcIJ}^{(+)} \quad (3.21)$$

Now we can use these new conical variables to define a new action, an action that is now base on $\frac{1}{2}$ flat connections.

²¹ A^{IJ} is the configuration variable and P_{IJ}^a is the conjugate momentum

²²Note that $\mathbf{\Omega}$ is still a two-form because we are taking exterior derivative of functions, i.e. the coefficients of their respective forms. The result is a one-form, and a two-form is generated when you have the exterior product of two one-forms.

3.4 Hamiltonian Formalism of Relativity

Now that the action has been defined using $\frac{1}{2}$ flat connections we can now formulate a Hamiltonian system given in the language of gauge theory. However the theory is currently a Lagrangian one and therefore we must perform a Legendre transformation in order to calculate the Hamiltonian and hence derive the Hamiltonian theory.

In order to perform a Legendre transformation we must first introduce a foliation in space-time and we must fix our notation of time due to the nature of general relativity. Time is dependent on the geometry of space-time and as a result it is represented by a dynamical vector field, the integral curves of this vector field intersect the foliation only once. Lie derivatives with respect to the vector field will replace usual time derivatives.[\[11\]](#)

We decompose the time evolution vector field, T^μ , into its normal and tangential parts.

$$T^\mu = Nn^\mu + N^\mu \tag{3.22}$$

where n^μ is the normal basis²³, N is the lapse function and N^μ is the called the shift vector field. The lapse function ‘pushes’ time in the normal direction and the shift vector field ‘pushes’ time in the tangential direction. We can now use these functions to project out the tangential and normal components time in the tetrads, co-frame fields.

Now we can use n^μ and the spatial metric,

$$h^{\mu\nu} = g^{\mu\nu} - n^\mu n^\nu \tag{3.23}$$

to project out the parts of the fields that are normal and tangential to the foliation. This essentially splits the field into normal and tangential parts of time but this

²³Normal to the foliation.

does not change the action or any resultant physics as this is just a projection²⁴. We can also use these to give a more formal definition to the lapse functions and the shift vector field, [14]

$$N = g_{\mu\nu} t^\mu n^\nu \quad (3.24)$$

$$N^\mu = h^\mu{}_\nu t^\nu \quad (3.25)$$

since $N^\mu n_\mu = 0$.

The spatial metric acting on the co-tetrads can be incorporated into the conjugate momentum, P_{IJ}^a , and as a result the conjugate momentum is based on the spatial projection of space-time. It is related by,

$$-\text{Tr}(P^a P^b) = P_{IJ}^a P^{bIJ} = \frac{1}{\kappa^2} (\det h) h^{ab} \quad (3.26)$$

where h^{ab} the 3-metric.

Now we can construct the action in terms of the Hamiltonian variables,

$$S = \int dt \int_M d^3x (P_{IJ}^a \mathcal{L}_T A_a^{IJ} - \mathcal{H}_{(+)}) \quad (3.27)$$

where the Hamiltonian density, $\mathcal{H}_{(+)}$, is defined as,

$$\mathcal{H}_{(+)} = - \left(\omega_{IJ}^{(+)} \cdot t \right) G^{IJ} + N^a C_a^{(+)} + N C_{(+)} \quad (3.28)$$

with,

$$G^{IJ} = \mathcal{D}_a P_{IJ}^a = \partial_a P_{IJ}^a + A_{aI}^K P_{JK}^a + A_{aJ}^K P_{IK}^a \quad (3.29a)$$

$$C_a^{(+)} = P_{IJ}^b F_{ab}^{IJ} \quad (3.29b)$$

²⁴This is quite similar to projecting out a vector, we are changing the basis into parts that are tangential to time and normal to time, this can only be done because we have expressed time as a dynamical vector field.

$$C_{(+)} = -\frac{\kappa}{\sqrt{|\det q|}} P_J^{aJ} P_J^{bK} F_{abK}^I \quad (3.29c)$$

where F_{ab}^{IJ} is the curvature of A_a^{IJ} , defined by, $F = dA + A \wedge A$.

Note that $\omega_{IJ}^{(+)} \cdot t$, N^a and N are the Lagrange multipliers and variations of the action with respect to these multipliers yields the following constraints,

$$G^{IJ} = 0; \quad C_a^{(+)} = 0; \quad \text{and} \quad C_{(+)} = 0 \quad (3.30)$$

Due to the use of $\frac{1}{2}$ -flat connection we find that we have casted general relativity as a gauge theory and as a result it contains a group structure which is similar to $SU(2)$ ²⁵ and A_a^{IJ} plays the role of the $SU(2)$ connection. [4]

The constraints in the equations (3.29) generate transformations in the groups structure:

- G^{IJ} generates internal gauge transformations.
- $C_a^{(+)}$ generates diffeomorphism on M .
- $C_{(+)}$ generates evolutions along Nn^μ .

These constraints become important during quantisation the gauge and diffeomorphism constraints are kinematical gauge symmetries and operate at fixed time hence kinematical. However the last constraint is the Hamiltonian constraint and it is used to describe the dynamics of the system.

3.5 Gauge Fixing

It is convenient to perform a partial gauge fixing. We define an internal vector field, $n^I \in M$, then we can define a projection operator, q_i^{I26} , on to the sub-space of M

²⁵ $SU(2)$ is for the Riemannian case, which we are deriving, however the group structure is actually the spin group $SO^+(\eta)$ (which is isomorphic to $SU(2)$).

²⁶ i, j, \dots represent spatial indices

that is orthogonal to n^I . This vector field is used to define the unit normal to the foliation,

$$n^\alpha = n^I e_I^\alpha \quad (3.31)$$

Then using the projection operator we can define the fixed metric on M ,

$$\eta_{ij} = q_i^I q_j^J \eta_{IJ} \quad (3.32)$$

Now the internal group structure is now defined by the subgroup based on η_{ij} , which leaves n^I invariant. Now each co-tetrad defines a co-triad, which is the projection of the tetrad onto M ,

$$e_a^i = e_\alpha^I q_i^J q_a^\alpha \quad (3.33)$$

the induced metric on M is given by,

$$q_{ab} = e_a^i e_b^j \eta_{ij} \quad (3.34)$$

Then the half flat connected one-form, $\omega_\alpha^{IJ(+)}$, on \mathcal{M} defines a connected one-form, Γ_a^i , on M , through the equation,

$$\partial_a e_b^j - \Gamma_{ab}^c e_c^j + \epsilon_{jkl} \Gamma_a^k e_b^l = 0 \quad (3.35)$$

where $\epsilon_{ijk} = q_i^I q_j^J q_k^K n^L \epsilon_{LIJK}$. $\omega_\alpha^{IJ(+)}$ also defines extrinsic curvature,

$$K_a^i = q_I^i q_a^\alpha \omega_\alpha^{IJ(+)} n_J \quad (3.36)$$

then the conjugate momentum on M can be expressed as,

$$P_i^a = \frac{1}{2\kappa\gamma} e_b^j e_c^k \eta^{abc} \epsilon_{ijk} \quad (3.37)$$

and the configuration variable can also be re-defined as,

$$A_a^i = \Gamma_a^i - \sigma\gamma K_a^i \quad (3.38)$$

where σ is the signature of the fixed metric, η_{ij} . In Lorentzian general relativity the connection is complex valued and the structure group is non-compact this creates a few problems and in order to resolve these problems was to do a partial gauge fixing. The result is the projection on to the spatial part of \mathcal{M} , M . As a result the structure group becomes compact and the connection is also real valued, only if we deal with real values of γ . Unfortunately this means that the expressions for the constraints and the Poisson algebra become more complex,[\[4\]](#)

$$G^i = \mathcal{D}_a P_i^a = \partial_a P_i^a + \epsilon_{ij}^k A_a^j P_k^a \quad (3.39a)$$

$$C_a = P_i^b F_{ab}^i - \frac{\sigma - \gamma^2}{\sigma\gamma} K_a^i G_i \quad (3.39b)$$

$$C = -\frac{\kappa\gamma^2}{2\sqrt{|\det q|}} P_i^a P_j^b \epsilon_k^{ij} F_{ab}^k + (\gamma^2 - \sigma)\kappa\partial_a \left(\frac{P_i^a}{\sqrt{|\det q|}} \right) G^i \quad (3.39c)$$

It is important to note the difference between A_a^{IJ} and A_a^i . A_a^{IJ} is the pull back to M of the space-time connection A_α^{IJ} , where as A_a^i is the connection defined by the spin connection and extrinsic curvature. The extrinsic curvature is the spatial projection of $\omega_\alpha^{IJ(+)}$ and the spin connection is defined through the spatial triad. [\[27\]](#)

The configuration variable A_a^{IJ} has , $3 \times 3 = 9$ degrees of freedom (3 internal and 3 space) and equations in [\(3.29\)](#) gives us $1 + 3 + 3 = 7$ constraints. This leaves 2 degrees of freedom, which variations of the action with respect to, A_a^{IJ} , and, P_{IJ}^a , yield the 2 equation of motion to take up the remaining 2 degrees of freedom.

3.6 The AMD Formalism

For the last 35 years efforts to formulate quantum general relativity has been slowed down by the Arnowitt-Deser-Misner (ADM) formalism, in particular the lack of mathematics on quantising the ADM formalism. The $3 + 1$ split was introduced in the ADM method however the variables used in the formalism was the metric and its corresponding extrinsic curvature and as a result the number of degrees of freedom did not balance out. However due to Ashtekar reformulating the connection theory of gravity has lead him and others to express general relativity as a gauge theory, and mathematical techniques in quantising a gauge theory have been around since Dirac and therefore it has lead to a steady progress in this field.

4 Refined Algebraic Quantisation

Refined algebraic quantisation was originally considered by Dirac and then generalised by Giulini and Marolf. [16] [15]

Refined algebraic quantisation allows quantisation of a Hamiltonian system with only first class constraints²⁷. Now the main idea of quantisation is that certain equations of motion (the constraints of a gauge system) need not be imposed directly at the operator level, but should be imposed as conditions that select certain physical states.[15]

The starting point is to have a classical system with a constrained phase space, Γ , and a symplectic one-form, ω , on Γ , which defines a Poisson bracket on the phase space. In order to quantise such a system there is a method based on the main ideas of [15] and summarised in [17] and for completeness the main points are listed below.

4.1 9 Steps to Algebraic Quantisation

1. Phase Space and Constraints

To begin we require a system defined by a Hamiltonian, H , with its set of first class constraints and a phase space is also required.

2. Polarisation

In order to quantise the phase space we must choose a suitable polarisation, we have to choose a sub-space known as the configuration space. This configuration space must have the following properties:

- (a) The configuration space, \mathcal{A} , should be large enough so that any function that is described on the space may be represented by \mathcal{A} elements.

²⁷General relativity has already been reformulated into a Hamiltonian system in section 3 but it is also only has first class constraints due to the equations in (3.30) are 0. i.e. the Poisson brackets between the constraints vanish.

- (b) The coordinates of \mathcal{A} must have a vanishing Poisson brackets.
- (c) \mathcal{A} should be closed under complex conjugation. [16]

The polarisation is how we split the symplectic manifold into the configuration variable and the momentum variable. Some choices of the variables make the equations of motion much simpler therefore our choice of polarisation will mean that our choice of phase space variables would yield the simpler equations of motion.

3. Poisson Sub-Algebra

We must define the sub-algebra by defining a space of smooth function, $C^\infty(\mathcal{C})$, and a space of smooth vector fields, $\mathcal{V}^\infty(\mathcal{C})$, on \mathcal{C} . The pair $C^\infty(\mathcal{C}) \times \mathcal{V}^\infty(\mathcal{C})$ forms a lie sub-algebra defined with elements of the functions, f , and vectors, v . The sub-algebra is denoted, \mathcal{B} .

4. Representation Theory of *-Algebra

We consider irreducible representations of the Poisson sub-algebra as linear operators on the kinematical Hilbert space. We define the quantum configuration space based on graphs, then consider its representation on the Hilbert space. This can only be done by define a measure on the Hilbert space which is an invariant volume element. This is to give our Hilbert space background independence.

5. Kinematical Representation

We chose suitable kinematical representation of the Hilbert space, that is the construction of the constraint operators on a dense domain with in the kinematical Hilbert space. The operators in this dense domain and their adjoints leave this space invariant, it is typically a space of smooth functions which are defined by the configuration variables.

6. Constraints

Then the constraint operators are used to generate gauge invariant states hence we impose them on the quantum system.

4.2 Construction of Poisson Sub-algebra

4.2.1 Polarisation

We have already defined in section 3 the Hamiltonian and the set of first class constraints. The variables A_a^i and P_i^a are used to define the phase space, where A_a^i is the $SU(2)$ connection in M and P_i^a is a vector density over M . These variables are our current choice of polarisation. The configuration space is, \mathcal{A} , which is the space of all suitably regular $SU(2)$ connections on M . [17]

4.2.2 Smearing Functions

Now we have chosen the polarisation we can go onto define the Poisson sub-algebra, to do this we must define a space of smooth function, $C^\infty(\mathcal{A})$, and smooth vectors, $V^\infty(\mathcal{A})$, on the configuration space, \mathcal{A} . At the moment it is not clear as to what smooth or differentiable means on the configuration space, as a result the configuration space must be given some form of manifold structure. A suitable choice would be to model the manifold structure on a Banach space²⁸, mainly due to its relation to Hilbert spaces and it makes an easier transition from the configuration space to the Hilbert space. However this creates a problem by use of this definition due to the configuration variable no longer being diffeomorphic, and this is because of its functional derivative²⁹ is proportional to a δ function. In order to preserve

²⁸A Banach space are defined as complete normed vector spaces.[18]

²⁹A function is differentiable at a point in \mathcal{A} if there exists a bounded linear function, which is nothing more than a functional derivative.

homeomorphism and diffeomorphism A_a^{IJ} must be smeared with a test function:

$$F[A] = \int d^3x F_{IJ}^a(x) A_a^{IJ} \quad (4.1)$$

where F_{IJ}^a is a smooth test function. Unfortunately this functions makes it difficult to construct $SU(2)$ invariant functions from them, but the theory is very similar to the canonical quantum Yang-Mills gauge theory and as a result a unques solution was found using Holonomy algebra. [17]

4.2.3 Holonomies

If we have a Riemannian manifold \mathcal{M} with a connection, ∇ , we can consider a set of closed curves, c , at a point, p , and the parallel transport of a vector along those closed curves. We find that the initial vector is not the same as the final vector, as a result we can define a map or a group of maps that transform the initial vectors to the final vectors along the closed curves. The set of transformation is called the holonomy group. The reason for the transformation is due to the curvature is the manifold³⁰ and the holonomy essentially describes the curvature of the connection, which means that it is background independent.[12]

In order to use holonomy we must first specify a principle fiber bundle³¹ to which the one-form, $A(c)$, which is a pull-back of a tensor on the fiber bundle to M . Now on bundles holonomy can be used to define group elements using the connected one-form, $A(c)$, as a result the equation is obtained.

$$g_c = \mathcal{P}e^{(-\oint_c A(c)dx)} \quad (4.2)$$

³⁰Consider a vector on a sphere, then move it along the sphere in any direction. Depending on the path the vector has been parallel transported the new vector will not be the same as the one you started with.

³¹see section 4.3.1 for definition.

where \mathcal{P} denotes path ordering. The trace of the integral yield Wilson loops functions, which are gauge invariant functions if c is a closed curve. It is this use of holonomies in the theory is the reason why quantum general relativity is labeled Loop Quantum gravity.

Holonomy is the geometrical consequence of curvature of the connection and as a result we can see how these functions change within the manifold, M . Given a curve, c from point t_1 to t_2 then the parallel transport is defined by the differential equation.

$$\frac{dh_{c,A}(t)}{dt}h = h_{c,A}(t)A_i^a(c(t))\frac{\tau_j}{2}\dot{c}^a(t) \quad (4.3)$$

where $A \in \mathcal{A}$, $h_{c,A}$ are elements of $SU(2)$ ³², τ_j are the $SU(2)$ generators and $A_i(c(t))$ is the connection. Due to $h_{c,A}$ being a holonomy it has the initial condition $h_{c,A}(0) = \mathbb{1}$. The unique solution to equation (4.3) is,

$$A(c) = h_{c,A}(1) \in SU(2) \quad (4.4)$$

The holonomy at point, $A \in M$, along a curve, c , is $A(c)$ which corresponds to an $SU(2)$ element $h_{c,A}(1)$ where $h_{c,A}$ is a set of functions that map the curve to $SU(2)$ elements.

We can see from equation (4.2) that holonomy actually smears³³ A_a^i only in one dimension, but in order to obtain a well defined Poisson algebra we require smooth functions of A_a^{IJ} . However this is not necessary if we smear the conjugate momentum variable as well.[17]

³²As defined by the The Berger classification

³³Smearing is a process used to remove delta functions to ensure that the connection is differentially smooth.

4.2.4 Smearing Fields

If we smear the conjugate momentum variable³⁴ then we must do so along two dimensions in order to ensure that Poisson algebra does not depend on a δ -distribution function. In order to smear the function we take its dual,

$$\Sigma = \epsilon P = *P \tag{4.5}$$

and integrate over a surface, S ,

$$E(S) = \int_S f_i \Sigma^i \tag{4.6}$$

where S is a two-dimensional open surface.

Now if we continue to calculate the naive Poisson bracket between momentum variables with respect to two smearing fields, we find that it is not zero.

$$\{P[S, f], P[S, g]\} \neq 0 \tag{4.7}$$

This is to ensure that the Jacobi identity between $A(e)$, $P[S, f]$ and $P[S, g]$ is equal to 0. In order to get round this problem we have to relate the momentum variable to a smooth vector field, $\mathcal{V}^\infty(\mathcal{C})$, by doing so the Poisson bracket becomes well defined³⁵. [21]

Since the configuration space has the group structure of $SU(2)$ and it's phase space defined as a co-tangent bundle. That means that every smooth function, f , on the configuration space defines a configuration variable on the phase space. Also every smooth vector field, X^i , on the configuration space defines a momentum

³⁴In the Yang-Mills theory this is denoted as electric field

³⁵In fact this was only considered while imposing refined algebraic quantisation on the theory

variable on the phase space,

$$P_X = X^i p_i \quad (4.8)$$

On a co-tangent bundle the Poisson brackets between the configuration and momentum variables mirrors the action of the vector field on the function. [4]

$$\{P_X, f\} = -\mathcal{L}_X f \quad \text{and} \quad \{P_X, P_Y\} = -P_{[X, Y]} \quad (4.9)$$

Now the lie algebra of functions and vector fields on the configuration space is now defined and as a result the lie algebra between the momentum and configuration observables are also defined. As we move to the quantum theory we find that configuration operators are associated with the smooth functions, \mathcal{C}^∞ , in the configuration space and the momentum operators are associated with the smooth vector fields, \mathcal{V}^∞ , also in the configuration space. The lie bracket between functions and vector spaces is defined as:

$$[(f', P_X), (f', P_{X'})] = (P_{X'} \cdot f - P_X \cdot f', [P_{X'}, P_X]) \quad (4.10)$$

The phase space will consist of the conjugate momentum and the configuration variable (A_a^i, P_i^a) where $A_a^i \in \mathcal{A}$ is the $\frac{1}{2}$ -flat connection it is smeared along the curve, e , to become the configuration observable $A(e) \in \mathcal{C}^\infty$, which is the holonomy of the connection along the curve, e .

P_i^a is the conjugate momentum variable and it is analogous to the Yang-Mills electric field. This is smeared along a surface, S , with a continuous function, f_i to give the momentum observable $P[S, f_i] \in \mathcal{V}^\infty$, the momentum observable is defined on the configuration space as a vector field.

The Poisson brackets between the configuration observables vanishes amongst themselves, this is due to the configuration space being defined as a co-tangent

bundle. The Poisson bracket between the configuration observable, $A(e)$, and the momentum observable, $P[S, f_i]$, is given as,

$$\{A(e), P[S, f_i]\} = \kappa \int d^3 f_a^i(x) \frac{\delta A(e)}{\delta A_a^{IJ}(x)} \quad (4.11)$$

which is given in [21] as,

$$\{Ae, P[S, f_i]\} = -\frac{\kappa}{2} f^i(q) \text{Tr} \left[A\left(\frac{1}{2}, \frac{1}{2}\right) \tau^i \right] + \frac{\kappa}{2} f^i(p) \text{Tr} [\tau^i A(1, 0)] \quad (4.12)$$

where p and q are where the surface, S , and the curve, e , intercept. $A(\frac{1}{2}, \frac{1}{2})$ denote the start and end of the closed curve. This is given for the example in the paper however it can be seen here that a generalised and more simplified version can be defined in [4] as,

$$\{A(e), P[S, f_i]\} = -\left[\frac{\kappa(S, e)}{2} \right] \times \begin{cases} A(e) \tau^i f_i(p) & \text{if } p \text{ is the source of } e \\ -f_i(p) \tau^i A(e) & \text{if } p \text{ is the target of } e \end{cases} \quad (4.13)$$

where the function $\kappa(S, e)$ denotes the intersection of e and S ,

$$\kappa(S, e) = \begin{cases} 0, & \text{if } e \cap S \text{ or } e \cap S = e \\ \kappa, & \text{if } e \text{ lies above } S \\ -\kappa, & \text{if } e \text{ lies below } S \end{cases} \quad (4.14)$$

We have smeared the conjugate momentum in two dimensions but can we do it in 3 dimensions? Unfortunately this is not the case due to an integral over the the extra dimension and therefore the algebra would not be closed. [17]

4.3 Representation of *-Algebra

We find that during the quantisation process our Hilbert space, the representation of our algebra, can be decomposed into Spin states, this is due to defining the momentum variable as a vector field in the configuration space, equation (4.8). As a result the vector field can then be defined as a left invariant or a right invariant vector field³⁶. Then the corresponding momentum functions can be defined in terms of the left and right invariant vector fields. These functions are just generalised angular momentum functions in $SU(2)$ and as a result the Hilbert space exhibits a spin network decomposition. [4]

4.3.1 Gauge Theory on Connections

Before we consider the decomposition of the Hilbert space we must ensure that the theory is gauge invariant, this means that the configuration space must be gauge invariant in order to generate the Hilbert space.

If M is our manifold then we can define a principle bundle³⁷ $B(M, G)$ over M . There exist a vertical subspace $V_u B$ which is a subspace of the tangent space $T_u B$ which is tangent G_p at u , where $u \in B(M, G)$ and p is its projection from $B(M, G)$ to M . We denote $V_u B$ as \mathcal{G} , the group of local gauge transformations. Then the quotient space \mathcal{A}/\mathcal{G} serves as the physical configurations space of classical gauge theory. [23]

4.3.2 Gauge Theory on Graphs

If we compare our theory to lattice gauge theory we are missing a very important object, the lattice! Now with hindsight we can see that our Hilbert space can be decomposed as a spin network decomposition; graphs are used to represent those

³⁶This is due to the vector field being defined on the configuration space which is a lie group

³⁷A principle bundle has a fiber bundle which is identical to a structure group G . [12]

spin states.

A graph, γ is a set of edges and vertices and has the following properties³⁸:

1. every edge $e \in \gamma$ is diffeomorphic with closed interval $[1, 0]$
2. the intersection of two different edges $e_1 \cap e_2$ is a vertex.
3. every edge is connected at at both ends with another element of, γ ³⁹.

By definition of a graph we can see that it also generates our ‘lattice’ in the comparison with lattice gauge theory, where each vertex represents a point or node and each edge is equivalent to the lattice spacing.

A graph, γ' can be larger than another graph, γ if every edge, e , can be written as a union of several edges, e' . It is this property that will create some difficulty when constructing the Hilbert space.

By restricting the bundle $B(M, G)$ to be a bundle over a graph, γ , then the B_γ becomes the space for γ , and we can further restrict the bundle over each edge, e , in each graph.

$$B_\gamma = \bigcup_{e \in \gamma} B_e \quad (4.15)$$

Then for every edge there exist a smooth connection A_e on B_e , if e is oriented then A_e is the parallel transport over the edge.[23]

We can restrict the configuration space, \mathcal{A} , to a configuration space over γ , \mathcal{A}_γ , then we can define a 1-1 map,

$$\pi_\gamma : \mathcal{A} \rightarrow \mathcal{A}_\gamma \quad (4.16)$$

This map can be used to define a projection between the configuration spaces between different graphs,

$$p_{\gamma\gamma'} : \mathcal{A}_\gamma \rightarrow \mathcal{A}_{\gamma'} \quad (4.17)$$

³⁸As defined in [23]

³⁹This is to ensure that the graph is closed.

Then we can take the projective limit of (4.17) by considering all graphs, this is denoted the quantum configuration space, $\bar{\mathcal{A}}$.

The connection, $A \in \mathcal{A}$, defines a point, $A_\gamma \in \mathcal{A}_\gamma$, then the collection of points, A_γ for all graphs defines a quantum connection, $(A_\gamma)_{\forall \gamma} = \bar{A}$, in the quantum configuration space, $\bar{A} \in \bar{\mathcal{A}}$. [23]

Since the configuration space, \mathcal{A} , is the space of smooth connections, A on the bundle, B , and by restricting the bundle we can restrict the configuration space and hence the connections to be smooth connections, $A_\gamma \in \mathcal{A}_\gamma$, on B_γ .

4.3.3 Quantum Configuration Space

The group for local gauge transformations is defined above and for each graph it is \mathcal{G}_γ , however we can restrict this to a subgroup, $\hat{\mathcal{G}}^\gamma$, where all local gauge transformations are the identity in B at the vertices of γ . Now the configuration space, \mathcal{A}_γ , is infinite dimensional due to redundancies when performing a local gauge transformation along the edges of the graph. [4] To remove these redundancies we must generate the quantum configuration space, $\bar{\mathcal{A}}_\gamma$, for a smooth quantum connection $\bar{A}_\gamma \in \bar{\mathcal{A}}_\gamma$. This can be done by defining $\bar{\mathcal{A}}_\gamma$ to be a quotient space,

$$\bar{\mathcal{A}}_\gamma = \mathcal{A}_\gamma / \hat{\mathcal{G}}^\gamma \quad (4.18)$$

By definition of a principle bundle [12] there exists a map between the bundle and its structure group, G , as a result there also is a map between the configuration space and the structure group, [23]

$$\Lambda_e : \mathcal{A}_e \rightarrow G \quad \text{and} \quad \Lambda_\gamma : \mathcal{A}_\gamma \rightarrow G^n \quad (4.19)$$

where n is the number of edges in the graph, γ . Every element, $A_e \in \mathcal{A}_e$, maps to a corresponding element in G , which is just the parallel transport, $A(e)$, along the

edge, e . By definition of quotient group there exist a projective map, [23]

$$p_\gamma : \mathcal{A} \rightarrow \mathcal{A}_\gamma / \widehat{\mathcal{G}}^\gamma \quad (4.20)$$

By combining with (4.19) we can define a surjective map between $\bar{\mathcal{A}}_\gamma$ and G^n [4]

$$p_\gamma^{-1} \cdot \Lambda_\gamma = \mathcal{I}_E : \bar{\mathcal{A}}_\gamma \rightarrow G^n; \quad \mathcal{I}_E(\bar{A}_\gamma) = (\bar{A}_\gamma(e_1), \dots, \bar{A}_\gamma(e_n)) \quad (4.21)$$

4.3.4 Hilbert Space

Given a smooth complex function, ϕ on G^n can use the inverse of the map in (4.19) to define a function Φ_γ on \mathcal{A}_γ .

$$\Phi_\gamma(A) = \phi(A(e_1), \dots, A(e_n)) \quad (4.22)$$

The space for such functions is denoted as Cyl_γ , and they are cylindrical functions over γ ⁴⁰. Combining this equation with the map (4.21) creates an isomorphism between the space for the cylindrical functions and its quantum counterpart, $\overline{\text{Cyl}}_\gamma$.

Now since the configuration space is modeled on a Banach space, and with the cyclic representation of the functions, the Hilbert space can be identified with a Lebesgue space; $\mathcal{H}_\gamma = L^2(\bar{\mathcal{A}}_\gamma, d\mu_\gamma^0)$, where μ_γ^0 is the measure on $\bar{\mathcal{A}}_\gamma$.

Now using the map (4.21) we can define functions, Ψ on the quantum configuration space, $\bar{\mathcal{A}}_\gamma$, as the pull back of the cylindrical functions, ψ , on G^n .

$$\Psi = \mathcal{I}_E^* \psi \quad (4.23)$$

Then functions on the quantum configuration space are states on the Hilbert space

⁴⁰These cylindrical functions that are the smooth functions required to define the sub-algebra from the configuration space, stated in point 3 of section 4.1

and as a result the inner product can be written as, [4]

$$\langle \Psi, \Phi \rangle = \int_{G^n} d\mu_H^0 \bar{\psi} \phi \quad (4.24)$$

where μ_H^0 is the Haar measure on G^n .

now we can extend this to the quantum configuration space, $\bar{\mathcal{A}}$, by taking the projective limit in section 4.3.2. By doing so the set of Haar measure, μ_H^0 , on $\bar{\mathcal{A}}_\gamma$ define a Borel measure, μ^0 , on $\bar{\mathcal{A}}$.

4.3.5 Spin Decomposition of Hilbert Space

Roger Penrose introduced the notion of a spin network decomposition, which was at the time his attempt to produce a combinatorial approach to space-time rather than that of manifolds. His definition involved the use of graph theory, which is equivalent to lattice gauge theory where the graph play the part of the lattice.[22]

Now we can decompose the Hilbert space as a sum of sub-spaces, which are based on each graph. But the Hilbert spaces for the graphs are not orthogonal to each other. As a result we can define a Hilbert space, \mathcal{M}'_γ , for each graph, which is orthogonal to a mutual Hilbert space, $\mathcal{H}_{\bar{\gamma}} = \mathcal{H}_{\bar{\gamma}'}$, where both Hilbert space are sub-spaces of \mathcal{H}_γ . [22]

Now the Hilbert space can be decomposed as,

$$\mathcal{H} = \oplus_\gamma \mathcal{H}'_\gamma \quad (4.25)$$

We define irreducible representations of G to the edges and vertices, where each is non-trivial; \mathbf{j}' as the set of edges in γ and \mathbf{l}' the set of vertices in γ . Then the Hilbert space is given by,

$$\mathcal{H}'_\gamma = \oplus_{\mathbf{j}', \mathbf{l}'} \mathcal{H}'_{\gamma, \mathbf{j}', \mathbf{l}'} \quad (4.26)$$

When the group $g = \text{SU}(2)$ then \mathbf{j} and \mathbf{l} become half-integers or spins, in that case we can define operators much like the angular momentum operators in quantum mechanics.

4.3.6 Kinematical Hilbert Space

The representation of the algebra is given as operators on the kinematical Hilbert space. Currently our Hilbert space is $\mathcal{H} = \oplus : \mathcal{H}_{\gamma, \mathbf{j}, \mathbf{l}}$, which can be further defined as $\mathcal{H}_\gamma = \oplus \mathcal{H}_{\mathbf{j}, \mathbf{l}}$.

Now we can define operators in the structure group and using the map (4.21) the Hilbert space can be decomposed to,

$$\mathcal{H}_\gamma = L^2(\bar{\mathcal{A}}_\gamma, d\mu_\gamma^0) = \otimes_n \mathcal{H}_\gamma^G = \otimes_n L^2(G, d\mu_H^0) \quad (4.27)$$

where n is the number of edges in the graph and $d\mu_H^0$ is the Haar measure on G . Then the operators defined on \mathcal{H}_γ^G can be naturally extended to operators on $\mathcal{H}_\gamma^A = L^2(\mathcal{A}_\gamma, d\mu_\gamma)$.

First the operators on the structure group are defined then it can be extended to the quantum configuration space by the maps previously defined.

A configuration variable is defined by a smooth function, $f_{(G)}$, on G then its corresponding operator is defined as,

$$\hat{f}_{(G)}\psi(g) = f_{(G)}(g) \psi(g) \quad (4.28)$$

where $g \in G$

The momentum operator, defined to each momentum function, is given as,

$$\hat{J}_{(G)}^{(X)}\psi(g) = i \left[\mathcal{L}_X \psi(g) + \frac{1}{2} (\text{div} X) \psi(g) \right] \quad (4.29)$$

where $\text{div}X$ is the divergence of X with respect to an invariant volume form. Then the commutators between the operators mirror their classical counterparts defined in (4.9). The momentum operator can be redefined to the right and left parts, when the vector field is either left invariant or right invariant to the structure group G .

$$\hat{L}_i = \hat{J}_i^{(L)} \quad \text{and} \quad \hat{R}_i = \hat{J}_i^{(R)} \quad (4.30)$$

Now using the map (4.21) we can extend these operators to the Hilbert space $\mathcal{H}_\gamma = L^2(\bar{\mathcal{A}}_\gamma, d\mu_\gamma)$.

$$\hat{J}_i^{(v,e)}\Psi = \mathcal{I}_E^* \left[\left(1 \otimes \dots \otimes 1 \otimes \hat{J}_i \otimes 1 \otimes \dots \otimes 1 \right) \psi \right] \quad (4.31)$$

where $\hat{J}_i^{(G)}$ is either left or right invariant if the vertex, v , is the source or target of the edge, e , respectively. The edge determines which space the operators act on and the vertex determines if the vector field is either left or right invariant. [4]

Now we can define the operators on the quantum configuration space. The configuration variables are represented by cylindrical functions, f on $\bar{\mathcal{A}}$, with corresponding operator \hat{f} ,

$$\hat{f}\psi(\bar{A}) = f(\bar{A}) \psi(\bar{A}) \quad (4.32)$$

because this operator act via multiplication it is similar to the operator acting on G . The the momentum operator, which are based on the momentum observable, is defined as,

$$\hat{P}_{(S,f)}\Psi(\bar{A}) = i\hbar \{P(S, f), \Psi(\bar{A})\} \quad (4.33)$$

where the bracket is defined as,

$$\{P(S, f), \Psi(\bar{A})\} = \sum_e \left[\frac{\partial \Psi(\bar{A}_e)}{\partial \bar{A}_e} \{P(S, f), \bar{A}_e\} \right] \quad (4.34)$$

where A_e is the holonomy along the edge and the bracket is defined in equation (4.13). If Ψ are cylindrical functions then the operator can be defined as,

$$\hat{P}_{(S,f)}\Psi(\bar{A}) = \frac{\hbar}{2} \sum_v f_i(v) \left[\sum_{eatv} \kappa(S, e) \hat{J}_i^{(v,e)} \Psi \right] \quad (4.35)$$

The momentum is a true derivative operator and only self-adjoint if the measure is invariant under translations in $SU(2)$. The operator defines an irreducible representation of the algebra on the kinematical Hilbert space \mathcal{H}_{kin} . [4]

5 Quantum Kinematics

5.1 Constraint Operators

Having now constructed a suitable representation of the kinematical Hilbert space we can apply the constraints and generate the corresponding operators that generate the symmetries.

5.1.1 Vector Constraint

Under the $SU(2)$ group transformations the spatial diffeomorphism can be defined as

$$\phi \cdot \bar{A}(e) = \bar{A}(\phi^{-1}(e)) \quad (5.1)$$

ϕ is a bundle for automorphism of the principle fiber bundle, B , and the equation ensure that the states, $\bar{A}(e)$, are invariant to these automorphisms, in the case that they are the states as Yang-Mill diffeomorphism. [24]

We can define a group, $\text{Diff}(M)$, as a group of analytical diffeomorphisms, and each element ϕ of this group naturally defines an isomorphism in the space Cyl . The operator that generates the spatial diffeomorphism can only do so if the measure is invariant to the diffeomorphism, then its map is,

$$\hat{V}_\phi : \text{Diff}(M) \rightarrow \mathcal{B}(\mathcal{H}_{kin}); \quad \phi \rightarrow \hat{V}(\phi) \quad (5.2)$$

and the operator is defined

$$\hat{V}_\phi \bar{\Psi}(\bar{A}(e)) = \bar{\Psi}(\phi \cdot \bar{A}(e)) = \bar{\Psi}(\bar{A}(\phi^{-1}e)) \quad (5.3)$$

$\text{Diff}(M)$ is a sub-space of all diffeomorphisms in the manifold, M , it allow a point to move around a neighborhood non-trivially (and trivially every where else in the

manifold). [4]

5.1.2 Gauss Constraint

Under the $SU(2)$ group transformations the gauge transformations can be defined as

$$\bar{g} \cdot \bar{A}(e) = g(b(e))\bar{A}(e) (g(f(e)))^{-1} \quad (5.4)$$

where $b(e)$ and $f(e)$ are the beginning and end of the edge respectively. \bar{g} is G -valued functions on M and exists within the group for the local gauge transformations $\bar{\mathcal{G}}$, defined by the quotient space,

$$\bar{\mathcal{G}} = \mathcal{G}/\mathcal{G}^0 \quad (5.5)$$

this is the group for the local gauge transformations in $\bar{\mathcal{A}}$.

The operator that generates the gauge transformations can only do so if the measure on $\bar{\mathcal{A}}$ is invariant under $\bar{\mathcal{G}}$, then its map is,

$$\hat{U}_{\bar{g}}: \bar{\mathcal{G}} \rightarrow \mathcal{B}(\mathcal{H}_{kin}); \quad \bar{g} \rightarrow \hat{U}(\bar{g}) \quad (5.6)$$

and the operator is defined

$$\hat{U}_{\bar{g}}\bar{\Phi}(\bar{A}(e)) = \bar{\Psi}(\bar{g} \cdot \bar{A}(e)) = \bar{\Psi}(g(b(e))\bar{A}(e) (g(f(e)))^{-1}) \quad (5.7)$$

So far the only state in \mathcal{H}_{kin} that is both gauge and diffeomorphism invariant is $\Psi(\bar{A}) = 1$, which can be thought of as the ground state. Since the momentum operator is a true derivative operator acting it on the ground state annihilates it. Elements of the Cyl represent excited ‘states’, because these excitations are only 1-dimensional then the quantum geometry is said to be polymer like. [4]

5.2 Quantum Kinematical Geometric Operators

All the geometrical operators that are defined are required to have the following basic definitions. [19]

1. The operator should be self-adjoint due to classical observables being real-valued
2. The operator should be well defined on a dense subspace, $\mathcal{D} = \text{Cyl}(\bar{\mathcal{A}})$ of \mathcal{H}_{kin}
3. The operator should be diffeomorphism covariant and independent of background fields used in regularisation.

Unfortunately due to time constraints only the area and volume operators are discussed here. The kinematical operators are not Dirac observables as they do not commute with the diffeomorphism operator. Their geometrical meaning of these operators is to parametrise the dense domain with specific eigenvalues of area and volume. This is defined thought matter which allows them to commute with the diffeomorphism constraint operator. [28] The background independence means that one cannot readily define geometrical objects, however we can construct them through the symmetries and the dynamics of the system.

5.2.1 Area Operator

When we smeared our momentum variables over an area, S , was never fully defined. S is a close finite surface without a boundary such that the closure of S in M is compact. Then the area of S can be well defined as a real-valued function on the full phase. The phase space is based on the triads, E_i^a ⁴¹ and hence given as,

$$A_S = \int_S dx^1 \wedge dx^2 [E_3^a E^{3a}]^{\frac{1}{2}} \quad (5.8)$$

⁴¹ $E_i^a = \text{sqrtdet}(q)e_i^a$

The problem here is that the expression involves square-roots of the triads which if we replace the triad with an operator valued distribution, (5.12), it would become badly divergent. Therefore we have to regularise this operator valued distribution by smearing it with respect to a function and split the integrand. [19]

The smearing of the triad has been done in section 4.2.4 but in order to split the integrand the test function must be 2 dimensional, therefore we define it as,

$$f_\epsilon(x, y) = \lim_{x \rightarrow \infty} \int_S d^2 y f_\epsilon(x^1, x^2; y^1, y^2) g(y^1, y^2) = g(x^1, x^2) \quad (5.9)$$

then in the limit that $\epsilon \rightarrow 0$ the test function tends to a delta function.

The smeared triad then becomes,

$$[E_i^3]_f(x) = \int_S d^2 y f_\epsilon(x, y) E_i^3(y) \quad (5.10)$$

The area operator can then be redefined in terms of the smeared triads.

$$[\hat{A}_S]_f = \int_S d^2 x \left[[E_i^3]_f(x) [E^{3i}]_f(x) \right]^{\frac{1}{2}} \quad (5.11)$$

To go over to the quantum theory we want to replace the triad with its operator valued distribution,

$$E_i^3 \rightarrow \hat{E}_i^3 = -i\hbar G \frac{\delta}{\delta A_a^i(x)} \quad (5.12)$$

This means that in order to ensure that the triad operator is well defined we would have to smear the connections, which we have already done in section 4.2.3. The holonomy for a smooth connection $A \in \mathcal{A}$ is equivalent to the group values for a generalised connection $\bar{A} \in \bar{\mathcal{A}}$, essentially given a generalised connection there exist a smooth connection for any given graph,

$$\bar{A}(e_k) = h_k[A] = \mathcal{P}e^{-\int_{e_k} A} \quad (5.13)$$

then this operator acting on the function, (4.21),

$$\left[\hat{E}_i^3 \right]_f(x) \cdot \Psi_\gamma(\bar{A}) = -iG\hbar \sum_{I=1}^N \int_S d^2y f_\epsilon(x, y) \left(\frac{\partial \bar{A}(e_k)}{\partial A_a^i(y)} \right) \left(\frac{\partial \psi}{\partial \bar{A}(e_k)} \right) \quad (5.14)$$

where,

$$\frac{\partial \bar{A}(e_k)}{\partial A_a^i(y)} = \int_0^1 dt \dot{e}_I^3(t) \delta(y^1, e_I^1(t)) \delta(y^2, e_I^2(t)) \delta(0, e_I^3(t)) (h_I(1, t) \tau^i h_I(t, 0)) \left(\frac{\partial \psi}{\partial \bar{A}(e_k)} \right) \quad (5.15)$$

Due to the delta distributions in the equation the operator becomes a distribution valued operator, to make it into a genuine operator we have to define a similar function to (4.14) in order to simplify the equation to consider those edges only intersect the surface, S ,

$$\left[\hat{E}_i^3 \right]_f(x) \cdot \Psi_\gamma = \frac{i\ell_p^2}{2} \sum_{I=1}^N \kappa_I f_\epsilon(x, e_I(0)) X_I^i \cdot \psi(\bar{A}(e_1), \dots, \bar{A}(e_N)) \quad (5.16)$$

where X is either a left invariant or right invariant vector field depending if the edges are outgoing or incoming respectively,

$$L_I^i \cdot \psi(\bar{A}(e_1), \dots, \bar{A}(e_N)) = (\bar{A}(e_I) \tau^i) \left(\frac{\partial \psi}{\partial \bar{A}(e_k)} \right) \quad (5.17)$$

$$R_I^i \cdot \psi(\bar{A}(e_1), \dots, \bar{A}(e_N)) = (-\tau^i \bar{A}(e_I)) \left(\frac{\partial \psi}{\partial \bar{A}(e_k)} \right) \quad (5.18)$$

Now we can use this to define the integrand,

$$\left[\hat{E}_i^3 \right]_f(x) \left[\hat{E}^{3i} \right]_f(x) \cdot \Psi_\gamma = -\frac{i\ell_p^4}{4} \left[\sum_{I,J} \kappa(I, J) f_\epsilon(x, v_{\alpha_I}) f_\epsilon(x, v_{\alpha_J}) X_J^j X_I^i \right] \Psi_\gamma \quad (5.19)$$

where $\left[\hat{E}_i^3 \right]_f(x) \left[\hat{E}^{3i} \right]_f(x)$ is defined to be the smeared operator of the determinant of the intrinsic metric on S , $[\hat{g}_S]_f(x)$. v_{α_I} is the vertex that the edge, e_I intersects

S and $\kappa(I, J)$ is then defined by,

$$\kappa(I, J) = \begin{cases} 0, & e_I \text{ and } e_J \text{ do not intersect } S \\ +1, & e_I \text{ and } e_J \text{ lies on the same side as } S \\ -1, & e_I \text{ and } e_J \text{ lies on the opposite sides as } S \end{cases} \quad (5.20)$$

As ϵ gets smaller we find that $f_\epsilon(x, v_{\alpha_I})f_\epsilon(x, v_{\alpha_J})$ is zero unless $v_{\alpha_I} = v_{\alpha_J}$. Since we defined, $\left[\hat{E}_i^3\right]_f(x)$, as a real valued self-adjoint operator hence its square root will be well defined. [19]

$$\widehat{\sqrt{g_S}}(x) = \frac{i\ell_P^2}{2} \sum_{\alpha} \delta^{(2)}(x, v_{\alpha}) \left[\sum_{I_{\alpha}, J_{\alpha}} \kappa(I_{\alpha}, J_{\alpha}) X_{J_{\alpha}}^j X_{I_{\alpha}}^i \right]^{\frac{1}{2}} \Psi_{\gamma} \quad (5.21)$$

Then plugging this in to the equation (5.8), the area operator becomes,

$$\hat{A}_S \cdot \Psi_{\gamma} = \frac{\ell_P^2}{2} \sum_{\alpha} \left[\sum_{I_{\alpha}, J_{\alpha}} \kappa(I_{\alpha}, J_{\alpha}) X_{I_{\alpha}}^i X_{J_{\alpha}}^j \right]^{\frac{1}{2}} \cdot \Psi_{\gamma} \quad (5.22)$$

We can define a vertex laplacian operator,

$$\Delta_{S, v_{\alpha}} = \kappa(I_{\alpha}, J_{\alpha}) X_{I_{\alpha}}^i X_{J_{\alpha}}^j \quad (5.23)$$

associated with a surface S and a vertex v , this is the laplacian operator on the kinematical Hilbert space and the area operator can be given as,

$$\hat{A}_S = \frac{\ell_P^2}{2} \sum_{v \in S} \sqrt{-\Delta_{S, v}} \quad (5.24)$$

The laplacian is defined in terms of the momentum operator defined in equation (4.31), which can itself be split into up and down operators,

$$-\Delta_{S, v} = \left(\hat{J}_{S, v}^{(d)i} - \hat{J}_{S, v}^{(u)i} \right)^2 = 2 \left(\hat{J}_{S, v}^{(d)i} \right)^2 + 2 \left(\hat{J}_{S, v}^{(u)i} \right)^2 - \left(\hat{J}_{S, v}^{(u+d)i} \right)^2 \quad (5.25)$$

the up and down operators identify those edges and are above or below the surface S . The eigenvalues of the laplacian can be calculated, which is trivial since the operators are angular momentum operators,

$$\lambda_{S,v} = 2j^d(j^d + 1) + 2j^u(j^u + 1) - 2j^{u+d}(j^{u+d} + 1) \quad (5.26)$$

hence the eigenvalues of the area operator, \hat{A}_S , are given by,

$$a_S = \frac{\ell_P^2}{2} \sum_{\alpha} [2j_{\alpha}^d(j_{\alpha}^d + 1) + 2j_{\alpha}^u(j_{\alpha}^u + 1) - 2j_{\alpha}^{u+d}(j_{\alpha}^{u+d} + 1)]^{\frac{1}{2}} \quad (5.27)$$

Then the ‘ground state’ can be calculated when $j^d = 0$, $j^U = \frac{1}{2}$ and $j^{d+u} = \frac{1}{2}$,

$$a_S = \frac{\ell_P^2}{2} \sqrt{j(h+1)} = \frac{\sqrt{3}}{4} \ell_P^2 \quad (5.28)$$

We see that the eigenvalues of the area operator become large very quickly and the spaces between each eigenvalue decreases exponentially, therefore it can be approximated to the continuum theory of physics. It is background independent therefore it is diffeomorphism covariant and it is gauge invariant.

5.2.2 Volume Operator

On Riemann structures the volume of a region, R , is given by,

$$V_R = \int_R d^3x \sqrt{q} = \int_R d^3x \sqrt{|\det E|} \quad (5.29)$$

where, R , is a fixed open region on M , also this region is based on a single coordinate system, x^a . The operator defined in the region can be summed over all regions in M regardless of the choice of R and how we split it.[20]

If we fix coordinates, x^a , in a region R and cover it with a set of cells, \mathcal{C} , and

each cell is oriented such that the sides are parallel to the coordinate planes. Then within every cube, C , has three 2-surfaces, $s = (S_1, S_2, S_3)$, then a partition, \mathcal{P} , of the region can be defined as the pair of cells and 2-surfaces, (C, s) . Then the triad can be smeared along each of these surfaces and the smeared metric is,

$$q_c[\tilde{E}] = \frac{1}{3!} \epsilon^{ijk} \eta_{abc} \tilde{E}_{S_a}^i \tilde{E}_{S_b}^j \tilde{E}_{S_c}^k \quad (5.30)$$

where,

$$\tilde{E}_S^i = \frac{1}{2} \int_S e_{ab}^i dx^a \wedge dx^b \quad (5.31)$$

and the determinant is given by $\frac{|q_c(t)|}{L_c^6}$, where L_c^6 is the size of the cell, and it is associated with a small positive ϵ such that the volume of the cell is less than ϵ . The volume of the region associated with a partition is given by,

$$V_R^{\mathcal{P}}[\tilde{E}] = \sum_{C \in \mathcal{C}} \sqrt{|q_c(\tilde{E})|} \quad (5.32)$$

Then as $\epsilon \rightarrow 0$ the volume of each cell does as well then,

$$V_R^{\mathcal{P}_\epsilon} \rightarrow V_R \quad (5.33)$$

then the metric operator of the smeared triads based on the cells is given by,

$$\hat{q}_c[\tilde{E}] = \frac{1}{48} \epsilon_{ijk} \eta_{abc} \sum_{x_1 \in S_a} \sum_{x_2 \in S_b} \sum_{x_3 \in S_c} \sum_{[e_1], [e_2], [e_3]} \kappa^a([e_1]) \kappa^b([e_2]) \kappa^c([e_3]) J_{x_1, e_1}^i J_{x_2, e_2}^j J_{x_3, e_3}^k \quad (5.34)$$

where $\kappa^d([e]) = \kappa_{S_d}$

In order to define the volume operator \hat{q}_c is required to be a self adjoint operator. Although J is a self adjoint operator the combination $J_{x_1, e_1}^i J_{x_2, e_2}^j J_{x_3, e_3}^k$ is not self-adjoint because the angular momentum like operators do not commute with each other. However due to the κ the operators for like edges are zero hence the equation

(5.34) is self adjoint because J commutes for distinct edges, therefore we can define a regularised volume operator as,

$$\hat{V}_R^{\mathcal{P}} = \sum_C |\hat{q}_C|^{\frac{1}{2}} \quad (5.35)$$

In the classical theory we can remove the regulator by taking the limit when $\epsilon \rightarrow 0$, however the limiting procedure cannot be done in the quantum theory because we have to define the restrictions on the partition, \mathcal{P} .

To remove the regulator we must first fix a graph on the region then allow the partition depend on the graph then take the limit, which works if each vertex of the graph is contained in the interior of each cell. The 2-surfaces $s = (S_1, S_2, S_3)$ cut the cube into two disjointed parts and then at the point at which all three 2-surfaces meet we define it as a vertex.[4] However if the graph does not intersect all the 2-surfaces then the cell does not have a vertex.

If we act the operator on a function, $\Psi_\gamma \in \text{Cyl}_{R(\gamma)}$, then if a cell does not contain a vertex then $\hat{q}_C \cdot \Psi_\gamma = 0$, this is due to the antisymmetry of η_{abc} . [20] Alternatively if a cell does contain a vertex then,

$$\hat{q}_c[\tilde{E}] = \frac{1}{48} \epsilon_{ijk} \epsilon_{abc} \sum_{I,J,K} \kappa^a([e_I]) \kappa^b([e_J]) \kappa^c([e_K]) J_{v,e_I}^i J_{v,e_2}^j J_{v,e_K}^k \quad (5.36)$$

where I, J, K label edges passing through the vertex. Then in the limit that $\epsilon \rightarrow 0$,

$$\epsilon_{abc} \kappa^a([e_I]) \kappa^b([e_J]) \kappa^c([e_K]) = \kappa([e_I], [e_J], [e_K]) \quad (5.37)$$

where $\kappa([e_I], [e_J], [e_K])$ vanishes if any of the edges lie in the same or opposite octants⁴². [20] The limiting procedure carries the memory of the partition, however this can be removed by averaging the regularised operators with respect to background

⁴²The splitting of a cube into eight equal parts around the center

structures, the resultant operator is q_c^{av} . In order to average this operator we must first consider a second coordinate system centered at the vertex. Then the action of a general linear group, $GL^+(3)$ ⁴³, relates the two coordinate system⁴⁴. The diagonal subgroup of *diag* rescales the coordinates leaving $\kappa^a [e_I]$ unchanged. Therefore to remove background dependence we consider the co-set space $GL_{(+)}(3)/diag$, denoted as \mathcal{S} , to average $\kappa [(e_I), (e_J), (e_K)]$.

To construct the background independent volume operator we have to average \hat{q}_c over the space \mathcal{S} . The second coordinate system $x_a(\theta)$, which is defined by the action of $GL^+(3)$ on $\theta^A \in \mathcal{S}$. Then we can construct the partition of the region based on these new coordinates, $\mathcal{P}(\theta)$, then the corresponding metric operator is \hat{q}_C^θ . These operators can be averaged on \mathcal{S} with a suitable probability measure on \mathcal{S} . Then we define a normalised function $\mu(\theta)$ such that,

$$\int_{\mathcal{S}} d^6\theta \mu(\theta) = 1 \quad (5.38)$$

Then the average is given by,

$$\hat{q}_C^{av} \cdot \Psi_\gamma = \int_{\mathcal{S}} d^6\theta \mu(\theta) \hat{q}_C^\theta \cdot \Psi_\gamma = \frac{1}{48} \sum_{I,J,K} \kappa^{av} \epsilon_{ijk} J_{v,e_I}^i J_{v,e_2}^j J_{v,e_K}^k \cdot \Psi_\gamma \quad (5.39)$$

where,

$$\kappa^{av} ([e_I], [e_J], [e_K]) = \int_{\mathcal{S}} d^6\theta \mu(\theta) \kappa ([e_I], [e_J], [e_K], \theta) \quad (5.40)$$

If there exist a normalised measure on \mathcal{S} then the κ^{av} must be of the type,

$$\kappa^{av} ([e_I], [e_J], [e_K]) = \kappa_\mu \epsilon([e_1], [e_2], [e_3]) \quad (5.41)$$

⁴³The (+) denotes the use of half-flat connections.

⁴⁴The second coordinate system must yield the same κ for all edges passing through the vertex

where $\epsilon([e_1], [e_2], [e_3])$ is the orientation function,

$$\epsilon([e_1], [e_2], [e_3]) = \begin{cases} 0, & \text{If the tangent directions are linearly dependent} \\ 1, & \text{If the tangent directions orientated positively} \\ -1, & \text{If the tangent directions orientated negatively} \end{cases} \quad (5.42)$$

where the tangent directions are oriented by a germ at e at a point x in M , so,

$$\int_S d^6\theta \mu(\theta) \kappa([e_I], [e_J], [e_K], \theta) = \kappa_0 \epsilon([e_I], [e_J], [e_K], \theta) \quad (5.43)$$

where κ_0 is a constant. With the average metric operator we can construct the average volume operator, \hat{V}_R^{av} ,

$$\hat{V}_R \cdot \Psi_\gamma = \kappa_0 \sum_v \sqrt{\hat{q}_v} \cdot \Psi_\gamma \quad (5.44)$$

where,

$$\hat{q}_v \cdot \Psi_\gamma = \frac{1}{48} \epsilon_{ijk} \sum_{e, e', e''} \epsilon(e, e', e'') J_{x,e}^i J_{x,e'}^j J_{x,e''}^k \cdot \Psi_\gamma \quad (5.45)$$

Then volume element operator is defined as,

$$\widehat{\sqrt{q(x)}} \cdot \Psi_\gamma = \kappa_0 \sum_v \delta^{(3)}(x, v) \sqrt{|\hat{q}_x|} \cdot \Psi_\gamma \quad (5.46)$$

when on \mathcal{H} ,

$$\hat{V}_R = \int_R d^3x \widehat{\sqrt{q(x)}} \quad (5.47)$$

By inspection we find that the volume operator is gauge invariant to diffeomorphism of M . So far in current research the eigenvalues and eigenvectors for the volume operator has only been calculated for specific cases,

6 Conclusions

As with any canonical formalism we expect the theory to give us a picture, or ideas to the physical implications to the theory rather than actual numerical calculations involved. So even though we have managed to define operators for area and volume but their physical meaning is not clear. The eigenvalues of the area operator (5.27) give the value for an area encompassed by all graphs. If we only consider one graph the the space of that graph occupies a volume and an area where the graph intersects the open 2-surface S , the area of intersection then takes discrete values. Essentially we are taking our system or space and calculating the area when it intersects another area that exists through the system/space. The larger the intersection the more values the area can take and we can see that the eigenvalues become large very quickly and hence can be approximated to the continuum theory nature of physics.

Analysis of the volume operator is much the same to that of the area operator however there is a major difference. The difference is how we measure the volume. With the area we have the open 2-surface and we consider the intersection of that to essentially measure the area spanned by the graph. The same should apply to volume operator as we should consider the intersection between the graph and an open 3-volume. The triads have been smeared two-dimensionally and cannot be smeared in three dimensions. So the alternative is to generate a volume from 3 2-surfaces and consider it intersection with the graph but then how should we interpret the corresponding eigenvalues? I believe that the solution lies in the smearing of the triads and, although we cannot smear in 3-dimensions, we can smear with respect to a function and an area. Then we might be able to use this to generate a volume and then construct a new volume operator.

It is important to note that this entire theory is based on the Hamiltonian formalism of loop quantum gravity, a formalism that involves a $3 + 1$ split of space-time. Although this is used to study the time evolution of a system it is in fact

that the evolution of time is dependent on the system. However our notion of time is part of the background geometry of space-time and as a result it will make it very difficult to construct a background independent theory and consider its time evolution. I believe the process may be easier, but more difficult conceptually, if we regard space-time as a moving entity. If space is moving at the same ‘rate’ as time, therefore we consider both evolutions in space-time rather than just focusing on the time evolution of a system. This may be the reason why physics breaks down at the Planck length because we consider the time evolution of a singularity rather than its space-time evolution.

The dynamics of quantum geometry provides a mathematical arena to formulate a non-perturbative dynamics of candidate quantum theories of gravity without any reference to a background. [26] Although not discussed in this report it is the main application for quantum geometry, we can construct a system and impose the Gauss and vector constraints and the system becomes gauge invariant and diffeomorphic. Then we could also impose the Hamiltonian constraint, which describes the time evolution of a background independent system. It is the dynamic of the theory that yield interesting results. Its applications in quantum cosmology and entropy calculations provide new avenues of discovery that were previously not considered. For a quick and insightful discussion read [26]

The questions quantum geometry should be able to answer can be summarised in [29]. It explains how quantum geometry can be used to explain, in slightly more detail, singularities and the big bang. Also the theory should be able to approximate to the continuous space-time at the macro level. As the space-time gets many orders of magnitude large than the Planck length the number of eigenvalues increases dramatically and the spaces between each eigenvalue decreases exponentially. As a result we can apply the continuum approximation early on, i.e. well before the macro level.

It is still difficult to conceptualise the discrete nature of the geometry and the role it plays in our universe, however these fundamental ideas open up an interesting area of physics namely background independence. An important step would be to construct background independent quantum field theory and analysis its dynamics. A few papers do attempt to construct a background independent scalar field [30] [31]. It would be interesting to see what type of predictions does make mainly regarding vacuum energy and its link to geometry.

References

- [1] Ashtekar, A. *Quantum Mechanics of Geometry* (1999) [gr-qc/990123v1]
- [2] Ashtekar, A. *Space and Time: From Antiquity to Einstein and Beyond* [<http://gravity.psu.edu/people/Ashtekar/articles/spaceandtime.pdf>]
- [3] Greene, B. *The Elegant Universe* (Vintage, 2005) [ISBN 0-09-928992-X]
- [4] Ashtekar, A; Lewandowski, J. *Background Independent Quantum Gravity: A Status Report* (2004) [gr-qc/0404018v2]
- [5] Davis, P.C.W; Betts, D.S. *Quantum Mechanics* (Nelson Thornes Ltd, 2002) [ISBN 0-7487-4446-0]
- [6] Zee, A. *Quantum Field Theory in a Nutshell* (Princeton University Press, 2003) [ISBN 0-691-01019-6]
- [7] Rajantie, A. *Advanced Quantum Field Theory* (2009) [Lecture Notes]
- [8] Holst, S. *Barbero's Hamiltonian derived from a generalised Hilbert-Palatini action* (1995) [gr-qc/9511026v1]
- [9] Mercuri, S. *Nieh-Yan invariant and Fermions in Ashtekar-Barbero-Immirzi formalism* (2007) [gr-qc/0610026v3]
- [10] Szekers, P. *A Course in Modern Mathematical Physics: Groups, Hilbert Spaces and Differential Geometry* (Cambridge, 2004) [ISBN 0-521-82960-7]
- [11] Ashtekar, A. *Lectures on non-perturbative Canonical Gravity* (World Scientific, 1991) [ISBN 981-02-0573-2]
- [12] Nakahara, M. *Geometry, Topology and Physics* (Talyor and Francis, 2003) [ISBN 0-7503-0606-8]

- [13] Kalinowski, M. W.; Piechicki, W. *Symplectic Structure for Field Theory* (Vol. 27 No. 10 pg. 2735, 1996) [<http://th-www.if.uj.edu.pl/acta/vol27/pdf/v27p2735.pdf>]
- [14] Calcagni, G.; Mercuri, S. *Barbero-Immirzi Field in Canonical Formalism of Pure Gravity* (2009) [gr-qc/0902.0957v2]
- [15] Marolf, D.; Giulini, D. *On the Generality of Refined Algebraic Quantization* (1999) [gr-qc/9812024v2]
- [16] Marolf, D. *Refined Algebraic Quantization: Systems with a Single Constraint* (1999) [gr-qc/9508015v3]
- [17] Thiemann, T. *Lectures on Loop Quantum Gravity* (2002) [gr-qc/0210094v1]
- [18] *Banach Spaces: Definition* [http://en.wikipedia.org/wiki/Banach_space]
- [19] Ashtekar, A; Lewandowski, J. *Quantum Theory of Geometry I: Area Operators* (1996) [gr-qc/9602046v2]
- [20] Ashtekar, A; Lewandowski, J. *Quantum Theory of Geometry II: Volume Operators* (1997) [gr-qc/9711031v1]
- [21] Ashtekar, A; Corichi, A; Zapata, J.A. *Quantum Theory of Geometry III: Non-commutativity of Riemannian Structures* (1998) [gr-qc/9806041v2]
- [22] Baez, J.C. *Spin Networks States in Gauge Theory* (1994) [gr-qc/9411007v1]
- [23] Ashtekar, A; Lewandowski, J. *Differential Geometry on the Space of Connections via graphs and Projective Limits* (1996) [gr-qc/9412073v2]
- [24] Lewandowski, J; Sahlmann, H; Okolów, A; Thiemann, T. *Uniqueness of diffeomorphism invariant states on holonomy - flux algebras* (2006) [gr-qc/0504147v2]
- [25] *The Planck Length* [<http://math.ucr.edu/home/baez/planck/node2.html>]

- [26] Ashtekar, A. *Gravity and the Quantum* (2004) [gr-qc/0410054v2]
- [27] Samuel, J. *Is Barberos Hamiltonian formulation a gauge theory of Lorentzian gravity?* (2000) [gr-qc/0005095v3]
- [28] Thiemann, T. *Introduction to Modern Canonical Quantum General Relativity* (2001) [gr-qc/0110034v1]
- [29] Ashtekar, A. *Quantum Space-times: Beyond the Continuum of Minkowski and Einstein* (2008) [0810.0514v2 [gr-qc]]
- [30] Kamiński, W; Lewandowski, J; Bobieński, M. *Background independent quantizations: the scalar field I* (2006) [gr-qc/0508091v2]
- [31] Kamiński, W; Lewandowski, J; Bobieński, M. *Background independent quantizations: the scalar field II* (2006) [gr-qc/0604112v2]